What is a Matrix?

A rectangular array of numbers, symbols, or characters assigned to a particular row and column is called a Matrix. The numbers, symbols, or characters present in the matrix are called elements of the matrix. The number of rows and columns present in a matrix determines the order of the matrix. For example if a matrix 'A' contains 'i' rows and 'j' columns then the matrix is represented by $[A]_{i\times j}$. Here, i×j determines the order of the matrix. Let us see an example of a matrix.

Determinants and matrices, in linear algebra, are used to solve linear equations by applying Cramer's rule to a set of non-homogeneous equations which are in linear form. Determinants are calculated for square matrices only. If the determinant of a matrix is zero, it is called a **singular determinant** and if it is one, then it is known as **unimodular**. For the system of equations to have a unique solution, the <u>determinant of the matrix</u> must be nonsingular, that is its value must be nonzero. In this article, let us discuss the definition of determinants and matrices, different matrices types, properties, with examples.

Types of Matrices

There are various types of matrices based on the number of rows and columns they have and also due to the specific characteristics shown by them. Let's see a few of them

Row Matrix: A Matrix in which there is only one row and no column is called Row Matrix.

Column Matrix: A Matrix in which there is only one column and now row is called a Column Matrix.

Horizontal Matrix: A Matrix in which the number of rows is less than the number of columns is called a Horizontal Matrix.

Vertical Matrix: A Matrix in which the number of columns is less than the number of rows is called a Vertical Matrix.

Rectangular Matrix: A Matrix in which the number of rows and columns are unequal is called a Rectangular Matrix.

Square Matrix: A matrix in which the number of rows and columns are the same is called a Square Matrix.

Diagonal Matrix: A square matrix in which the non-diagonal elements are zero is called a Diagonal Matrix.

Zero Matrix: A matrix whose all elements are zero is called a Zero Matrix.

Unit Matrix: A diagonal matrix whose all diagonal elements are 1 is called a Unit Matrix.

Symmetric matrix: A square matrix is said to be symmetric if the transpose of the original matrix is equal to its original matrix. i.e. $(A_T) = A$.

Skew-symmetric: A skew-symmetric (or antisymmetric or antimetric[1]) matrix is a square matrix whose transpose equals its negative.i.e. $(A_T) = -A$.

Types of Matrices

There are different types of matrices. Let's see some of the examples of different types of matrices

- Zero Matrix: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- Identity Matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Symmetric Matrix: $\begin{bmatrix} 2 & 3 & -1 \\ 3 & 0 & 6 \\ -1 & 6 & 5 \end{bmatrix}$
- Diagonal Matrix: $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
- Upper Triangular Matrix: $\begin{bmatrix} 6 & -1 & 5 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{bmatrix}$
- Lower Triangular Matrix: $\begin{bmatrix} 6 & 0 & 0 \\ 2 & 4 & 0 \\ 8 & -1 & 2 \end{bmatrix}$

Inverse of a Matrix

Inverse of a matrix is defined usually for square matrices. For every m × n square matrix, there exists an inverse matrix. If A is the square matrix then A^{-1} is the inverse of matrix A and satisfies the property:

$AA^{-1} = A^{-1}A = I$, where I is the Identity matrix.

Also, the determinant of the square matrix here should not be equal to zero.

Definition of Transpose of a Matrix

The Transpose of a matrix is a mathematical operation that involves flipping the rows and columns of the original matrix.

Representation of Transpose of Matrices

$$A = [a_{(ij)}]_{m \times n}$$
$$A^t = [a_{(ji)}]_{n \times m}$$

here i, j present the position of a matrix element, row- and column-wise, respectively, such that, $1 \leq i \leq m$ and $1 \leq j \leq n$.

Example: For any given matrix A of order 2 × 3 its transpose is?

$$A = \begin{bmatrix} 2 & 5 & 3 \\ 4 & 7 & 0 \end{bmatrix}$$

Solution:

Transpose of A

$$A^t = \begin{bmatrix} 2 & 4 \\ 5 & 7 \\ 3 & 0 \end{bmatrix}$$

*Order of A^t is **3 × 2***

Definition of Determinant

A determinant can be defined in many ways for a square matrix.

The first and most simple way is to formulate the determinant by taking into account the top row elements and the corresponding minors. Take the first element of the top row and multiply it by it's minor, then subtract the product of the second element and its minor. Continue to alternately add and subtract the product of each element of the top row with its respective minor until all the elements of the top row have been considered.

For example let us consider a 4×4 matrix A.

$$A = \begin{bmatrix} m & n & o & p \\ q & r & s & t \\ u & v & w & x \\ y & z & a & b \end{bmatrix}$$

Now its determinant |A| is defined as

$$|A| = \begin{vmatrix} m & n & o & p \\ q & r & s & t \\ u & v & w & x \\ y & z & a & b \end{vmatrix}$$

$$= m\begin{vmatrix} r & s & t \\ v & w & x \\ z & a & b \end{vmatrix} - n\begin{vmatrix} q & s & t \\ u & w & x \\ y & a & b \end{vmatrix} + o\begin{vmatrix} q & r & t \\ u & v & x \\ y & z & b \end{vmatrix} - p\begin{vmatrix} q & r & s \\ u & v & w \\ y & z & a \end{vmatrix}$$

If the determinant of a square matrix A is zero, then A is **not invertible**. In other words, the matrix A is **singular**. This is because the determinant of a matrix is a scalar value that represents the scaling factor by which the matrix transforms the area of a unit square to the area of a parallelogram formed by the vectors of the transformed unit square. If the determinant is zero, then the area of the parallelogram is zero, which means that the matrix transformation collapses the unit square to a line or a point. This makes it impossible to find the inverse of the matrix A, which is required to solve systems of linear equations.

What is **Invertible Matrix?** A matrix A of dimension n x n is called invertible if and only if there exists another matrix B of the same dimension, such that **AB = BA = I**, where I is the identity matrix of the same order. Matrix B is known as the inverse of matrix A. Inverse of matrix A is symbolically represented by A -1.

Question: If the value of a third order determinant is 11 then the value of the square of the determinant formed by the cofactors will be

Solution

Value of the determinant formed by cofactors = $11^{3-1} = 11^2 = 121$
Therefore, value of the square of determinant formed by cofactor = $(121)^2 = 14641$

How to Find the Cofactor?

Let's consider the following matrix:

$$\begin{bmatrix} 6 & 4 & 3 \\ 9 & 2 & 5 \\ 1 & 7 & 8 \end{bmatrix}$$

To find the cofactor of 2, we put blinders across the 2 and remove the row and column that involve 2, like below:

$$\begin{bmatrix} 6 & 3 \\ 1 & 8 \end{bmatrix}$$

Now we have the matrix that does not have 2. We can easily find the determinant of a matrix of which will be the cofactor of 2. Multiplying the diagonal elements of the matrix, we get.

- 6 x 8 = 48
- 3 x 1 = 3

Now subtract the value of the second diagonal from the first, i.e, 48 – 3 = 45.

Check the sign that is assigned to the number. Every 3 x 3 determinant carries a sign based on the position of the eliminated element.

The Matrix sign can be represented to write the **cofactor matrix** is given below-

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Check the actual location of the 2. You can note that the positive sign is in the previous place of the 2. Hence, the resultant value is +3, or 3.

Transpose of a Matrix Properties

Let's learn about the important properties of the transpose of a matrix:

- A square matrix "A" of order "n × n" is said to be an orthogonal matrix, if $AA^T = A^TA = I$, where "I" is an identity matrix of order "n × n."

- A square matrix “A” of order “n × n” is said to be a symmetric matrix if its transpose is the same as the original matrix, i.e., $A^T = A$.
- A square matrix “A” of order “n × n” is said to be a skew-symmetric matrix if its transpose is equal to the negative of the original matrix, i.e., $A^T = -A$.
- **Double Transpose of a Matrix:** Transpose of the transpose matrix is the original matrix itself.

$$(A^t)^t = A$$

- **Transpose of Product of Matrices:** This property says that

$$(AB)^t = B^t A^t$$

- **Multiplication by Constant:** If a matrix is multiplied by a scalar value and its transpose is taken, then the resultant matrix will be equal to the transpose of the original matrix multiplied by the scalar value, i.e., $(kA)^t = kA^t$, where k is a scalar value.

- **Transpose of Addition of Matrices:** This property says that.

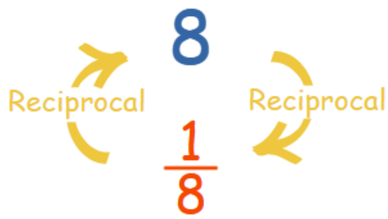
$$(A + B)^t = A^t + B^t$$

- If “A” is a square matrix of any order and is invertible, then the inverse of its transpose is equal to the transpose of the inverse of the original matrix, i.e., $(A^t)^{-1} = (A^{-1})^t$.

Inverse of a Matrix

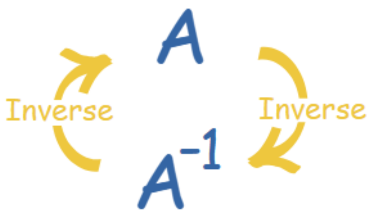
What is the Inverse of a Matrix?

Just like a **number** has a reciprocal ...



Reciprocal of a Number (note: $\frac{1}{8}$ can also be written 8^{-1})

... a **matrix** has an **inverse :**



Inverse of a Matrix

We write A^{-1} instead of $\frac{1}{A}$ because we don't divide by a matrix!

And there are other similarities:

When we **multiply a number** by its **reciprocal** we get **1**:

$$8 \times 18 = \mathbf{1}$$

When we **multiply a matrix** by its **inverse** we get the **Identity Matrix** (which is like "1" for matrices):

$$A \times A^{-1} = \mathbf{I}$$

Same thing when the inverse comes first:

$$18 \times 8 = \mathbf{1}$$
$$A^{-1} \times A = \mathbf{I}$$

Definition

Here is the definition:

The inverse of A is A^{-1} only when:

$$AA^{-1} = A^{-1}A = \mathbf{I}$$

Sometimes there is no inverse at all.

(Note: writing AA^{-1} means A times A^{-1})

2x2 Matrix

OK, how do we calculate the inverse?

Well, for a 2x2 matrix the inverse is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In other words: **swap** the positions of a and d, put **negatives** in front of b and c, and **divide** everything by **ad−bc** .

Note: **ad−bc** is called the determinant.

Let us try an example:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4\times6-7\times2}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

How do we know this is the right answer?

Remember it must be true that: $AA^{-1} = \mathbf{I}$

Why Do We Need an Inverse?

Because with matrices we **don't divide**! Seriously, there is no concept of dividing by a matrix.

But we can **multiply by an inverse**, which achieves the same thing.

Imagine we can't divide by numbers ...

... and someone asks "How do I share 10 apples with 2 people?"

But we can take the **reciprocal** of 2 (which is 0.5), so we answer:

10 × 0.5 = 5

They get 5 apples each.

The same thing can be done with matrices:

Say we want to find matrix X, and we know matrix A and B:

$$XA = B$$

It would be nice to divide both sides by A (to get X=B/A), but remember **we can't divide**.

But what if we multiply both sides by A^{-1} ?

$$XAA^{-1} = BA^{-1}$$

And we know that $AA^{-1} = I$, so:

$$XI = BA^{-1}$$

We can remove I (for the same reason we can remove "1" from 1x = ab for numbers):

$$X = BA^{-1}$$

And we have our answer (assuming we can calculate A^{-1})

In that example we were very careful to get the multiplications correct, because with matrices the order of multiplication matters. AB is almost never equal to BA.

A Real Life Example: Bus and Train

A group took a trip on a **bus**, at $3 per child and $3.20 per adult for a total of $118.40.

They took the **train** back at $3.50 per child and $3.60 per adult for a total of $135.20.

How many children, and how many adults?

First, let us set up the matrices (be careful to get the rows and columns correct!):

First, let us set up the matrices (be careful to get the rows and columns correct!):

$$\overset{\text{Child} \quad \text{Adult}}{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} \overset{\text{Bus} \quad \text{Train}}{\begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix}} = \overset{\text{Bus} \quad \text{Train}}{\begin{bmatrix} 118.4 & 135.2 \end{bmatrix}}$$

This is just like the example above:

$$XA = B$$

So to solve it we need the inverse of "A":

$$\begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix}^{-1} = \frac{1}{3 \times 3.6 - 3.5 \times 3.2} \begin{bmatrix} 3.6 & -3.5 \\ -3.2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 8.75 \\ 8 & -7.5 \end{bmatrix}$$

Now we have the inverse we can solve using:

$$X = BA^{-1}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 118.4 & 135.2 \end{bmatrix} \begin{bmatrix} -9 & 8.75 \\ 8 & -7.5 \end{bmatrix}$$

$$= \begin{bmatrix} 118.4 \times -9 + 135.2 \times 8 & 118.4 \times 8.75 + 135.2 \times -7.5 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 22 \end{bmatrix}$$

There were 16 children and 22 adults!

Order is Important

Say that we are trying to find "X" in this case:

$$AX = B$$

This is different to the example above! X is now **after** A.

With matrices the order of multiplication usually changes the answer. Do not assume that AB = BA, it is almost never true.

So how do we solve this one? Using the same method, but put A^{-1} in front:

$$A^{-1}AX = A^{-1}B$$

And we know that $A^{-1}A = I$, so:

$$IX = A^{-1}B$$

We can remove I:

$$X = A^{-1}B$$

And we have our answer (assuming we can calculate A^{-1})

Bigger Matrices

The inverse of a 2x2 is **easy** ... compared to larger matrices (such as a 3x3, 4x4, etc).

For those larger matrices there are three main methods to work out the inverse:

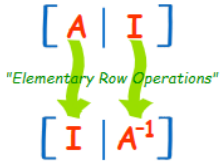
- Inverse of a Matrix using Elementary Row Operations (Gauss-Jordan)
- Inverse of a Matrix using Minors, Cofactors and Adjugate
- Use a computer (such as the Matrix Calculator)

Inverse of a Matrix using Elementary Row Operations

Also called the Gauss-Jordan method

This is a fun way to find the Inverse of a Matrix:

Play around with the rows (adding, multiplying or swapping) until we make Matrix **A** into the Identity Matrix **I**



And by ALSO doing the changes to an Identity Matrix it magically turns into the Inverse!

The **"Elementary Row Operations"** are simple things like adding rows, multiplying and swapping ... let's see with an example:

Example: find the Inverse of "A":

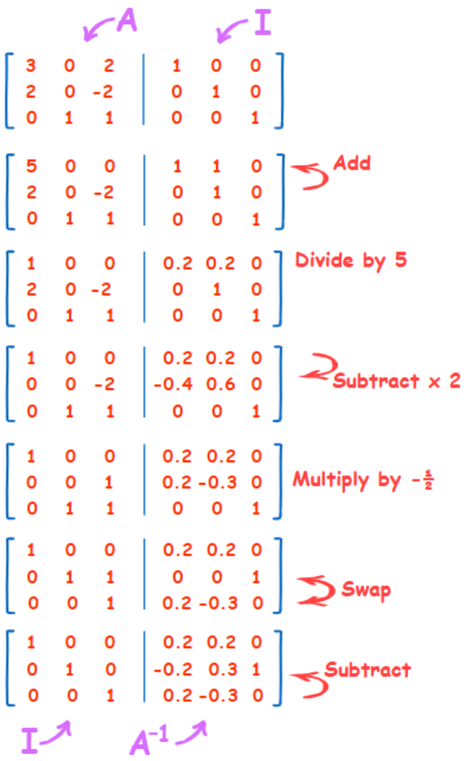
$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

We start with the matrix A, and write it down with an Identity Matrix I next to it:

But we can only do these **"Elementary Row Operations"**:

- **swap** rows
- **multiply** or **divide** each element in a a row by a constant
- replace a row by **adding** or **subtracting** a multiple of another row to it

And we must do it to the **whole row**, like in this example:



Start with **A** next to **I**

Let's add row 2 to row 1,

Then divide row 1 by 5

Then take 2 times the first row, and subtract it from the second row,

Multiply second row by -1/2,

Swap the second and third row,

Last, subtract the third row from the second row,

And we are done!

Matrix Multiplication Definition

Matrix multiplication, also known as matrix product and the multiplication of two matrices, produces a single matrix. It is a type of binary operation.

If A and B are the two matrices, then the product of the two matrices A and B are denoted by:

X = AB

Hence, the product of two matrices is the dot product of the two matrices.

Matrix multiplication Condition

To perform **multiplication of two matrices**, we should make sure that the number of columns in the 1st matrix is equal to the rows in the 2nd matrix. Therefore, the resulting matrix product will have a number of rows of the 1st matrix and a number of columns of the 2nd matrix. The order of the resulting matrix is the **matrix multiplication order**.

How to Multiply Matrices?

Let's learn how to multiply matrices.

Consider matrix A which is a × b matrix and matrix B, which is a b ×c matrix.

Then, matrix C = AB is defined as the A × B matrix.

An element in matrix C, C_{xy} is defined as:

$$C_{xy} = A_{x1}B_{y1} + \ldots . + A_{xb}B_{by} = \sum_{k=1}^{b} A_{xk}B_{ky}$$

For x = 1...... a and y= 1.......c

Notation

If A is a m×n matrix and B is a p×q matrix, then the matrix product of A and B is represented by:

X = AB

Where X is the resulting matrix of m×q dimension.

3×3 Matrix Multiplication

To understand the multiplication of two 3 × 3 matrices, let us consider two 3 × 3 matrices A and B.

$$A = \begin{bmatrix} 12 & 8 & 4 \\ 3 & 17 & 14 \\ 9 & 8 & 10 \end{bmatrix}, B = \begin{bmatrix} 5 & 19 & 3 \\ 6 & 15 & 9 \\ 7 & 8 & 16 \end{bmatrix}$$

Each element of the Product matrix AB can be calculated as follows:

- AB_{11} = 12×5 + 8×6 + 4×7 = 136
- AB_{12} = 12×19 + 8×15 + 4×8 = 380
- AB_{13} = 12×3 + 8×9+4×16 = 172
- AB_{21} = 3×5 + 17×6 + 14×7 = 215
- AB_{22} = 3×19 + 17×15 + 14×8 = 424
- AB_{23} = 3×3 + 17×9 + 14×16 = 386
- AB_{31} = 9×5 + 8×6 + 10×7 = 163
- AB_{32} = 9×19 + 8×15 + 10×8 = 371
- AB_{33} = 9×3 + 8×9 + 10×16 = 259

Therefore,

$$AB = \begin{bmatrix} 136 & 380 & 172 \\ 215 & 424 & 386 \\ 163 & 371 & 259 \end{bmatrix}$$

Rank of Matrix

The maximum number of linearly independent columns (or rows) of a matrix is called the **rank of a matrix**. The rank of a matrix cannot exceed the number of its rows or columns.

If we consider a square matrix, the columns (rows) are linearly independent only if the matrix is nonsingular. In other words, the rank of any nonsingular matrix of order m is m. The rank of a matrix A is denoted by ρ(A).

The rank of a null matrix is zero. A null matrix has no non-zero rows or columns. So, there are no independent rows or columns. Hence, the rank of a null matrix is zero.

How to Find the Rank of a Matrix?

To find the rank of a matrix, we will transform the matrix into its echelon form.

Then, determine the rank by the number of non-zero rows.

Consider the following matrix.

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 8 & 12 \end{bmatrix}$$

While observing the rows, we can see that the second row is two times the first row. Here, we have two rows, but it does not count. The rank is considered as 1.

Consider the unit matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can see that the rows are independent. Hence, the rank of this matrix is 3.

The rank of a unit matrix of order m is m.

If A matrix is of order m×n, then $\rho(A) \leq \min\{m, n\}$ = minimum of m, n.

If A is of order n×n and $|A| \neq 0$, then the rank of A = n.

If A is of order n×n and $|A| = 0$, then the rank of A will be less than n.

Rank of a Matrix by Row-Echelon Form

We can transform a given non-zero matrix to a simplified form called a Row-echelon form using the row elementary operations. In this form, we may have rows, all of whose entries are zero. Such rows are called zero rows. A non-zero row is one in which at least one of the elements is not zero.

A non-zero matrix A is said to be in a row-echelon form if:

(i) All zero rows of A occur below every non-zero row of A.

(ii) The first non-zero element in any row i of A occurs in the j^{th} column of A, and then all other elements in the j^{th} column of A below the first non-zero element of row i are zeros.

(iii) The first non-zero entry in the i^{th} row of A lies to the left of the first non-zero entry in $(i + 1)^{th}$ row of A.

Note: *A non-zero matrix is said to be in a row-echelon form if all zero rows occur as bottom rows of the matrix and if the first non-zero element in any lower row occurs to the right of the first non-zero entry in the higher row.*

Find the rank of the given matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

Solution:

Given,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

Now, we transform matrix A to echelon form by using elementary transformation.

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows = 2

Hence, the rank of matrix A = 2

Example:

Given

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

We get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, the number of non-zero rows = 1

Hence, the rank of the matrix = 1

Example 4:

Find the rank of the 2×2 matrix

$$B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Solution:

Given,

$$B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Order of B = 2×2

|B| = 40 – 42 = -2 ≠ 0

So, the rank of B = 2